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Weil's converse theorem with poles

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We prove a generalization of the classical converse theorem of Weil, allowing the twists by non-trivial Dirichlet characters to have arbitrary poles.

1 Introduction

In this paper we prove a direct generalization of Weil's converse theorem for classical holomorphic modular forms (see [12], [8, Thm 4.3.15]), improving on the method of [1]. Our precise result is the following:

Theorem 1.1. Let ψ be a Dirichlet character modulo N, k a positive integer satisfying $\psi(-1) = (-1)^k$, and $\{f_n\}_{n=1}^{\infty}$, $\{g_n\}_{n=1}^{\infty}$ sequences of complex numbers satisfying $f_n, g_n = O(n^\sigma)$ for some $\sigma > 0$. Let $\mathcal{P}$ be a set of prime numbers such that $\{p \in \mathcal{P} : p \equiv u \pmod{v}\}$ is infinite for every $u, v \in \mathbb{Z}_{>0}$ with $(u, v) = 1$, and $p \nmid N$ for any $p \in \mathcal{P}$. For every primitive Dirichlet character χ of modulus $q \in \mathcal{P} \cup \{1\}$, assume that the twisted L-functions

$$\Lambda_f(s,\chi) = (2\pi)^{-s}\Gamma(s)\sum_{n=1}^{\infty} f_n\chi(n)n^{-s} \text{ and } \Lambda_g(s,\overline{\chi}) = (2\pi)^{-s}\Gamma(s)\sum_{n=1}^{\infty} g_n\overline{\chi}(n)n^{-s},$$

defined initially for $\Re(s) > \sigma + 1$, continue to meromorphic functions on $\mathbb{C}$ and satisfy the functional equation

$$\Lambda_f(s,\chi) = \epsilon\psi(q)\chi(N)\frac{\tau(\chi)^2}{q}(q^2N)^{\frac{k}{2}-s}\Lambda_g(k-s,\overline{\chi}), \tag{1}$$

where $\tau(\chi) = \sum_{n=1}^{q}\chi(n)e(n/q)$ denotes the Gauss sum and $\epsilon \in \mathbb{C}^\times$ is fixed. Let $\Lambda_f(s) = \Lambda_f(s, \mathbf{1})$, where $\mathbf{1}$ denotes the character of modulus 1, and set

$$f_0 = -\operatorname{Res}_{s=0}\Lambda_f(s), \quad f(z) = \sum_{n=0}^{\infty} f_n e(nz).$$

Suppose that there is a non-zero polynomial $P \in \mathbb{C}[s]$ such that $P(s)\Lambda_f(s)$ continues to an entire function of finite order. Then

(i) if $k \neq 2$ or ψ is non-trivial then $f \in M_k(\Gamma_0(N), \psi)$;
(ii) if $k = 2$ and ψ is trivial then $f - cE_2 \in M_2(\Gamma_0(N))$, where $c = \frac{\pi}{6}\operatorname{Res}_{s=1}\Lambda_f(s)$ and $E_2(z) = 1 - 24\sum_{n=1}^{\infty}\frac{ne(nz)}{1-e(nz)}$ is the Eisenstein series of weight 2 and level 1.

$\square$

Remarks.

Received

1. If we define g similarly to f, namely

$$\Lambda_g(s) = \Lambda_g(s, \mathbf{1}), \quad g_0 = -\operatorname{Res}_{s=0} \Lambda_g(s), \quad g(z) = \sum_{n=0}^{\infty} g_n e(nz),$$

then for $k \neq 2$ or ψ non-trivial it follows that $g \in M_k(\Gamma_0(N), \overline{\psi})$ and $g(z) = \epsilon^{-1}(-i\sqrt{N}z)^{-k} f(-1/Nz)$. Moreover, if the series defining $\Lambda_f(s)$ converges absolutely at $s = k - \delta$ for some $\delta > 0$, then f and g are cusp forms. If $k = 2$ and ψ is trivial then similar remarks apply if one replaces f by $f - cE_2$.
2. The main improvement over Weil's theorem is that the twists $\Lambda_f(s, \chi)$ for non-trivial χ may be arbitrary meromorphic functions, subject only to the functional equation (1). Theorem 1.1 also generalizes a result of Weissauer [13], who obtained a similar conclusion under the assumption that every twist has at most finitely many poles; see also [5] and [10] for the special case $N = 1$.

□

We follow the basic method of [1, §2]. The result given there assumed stronger analytic properties for a larger set of twists. As the proof of Theorem 1.1 will show, we can deduce the extra properties in the course of the proof, starting from only weak analytic properties for the same set of moduli q as in Weil's original converse theorem [12]. Note that Weil's hypothesis has since been substantially weakened by various authors. We mention in particular works of Razar [11] and Khoai [6], who gave classical versions of the theorems of Piatetski-Shapiro [9] and Li [7]; these imply that it is enough to assume the functional equation (1) for an explicit *finite* set of moduli q (depending on N). It is possible that the improvements given by these versions could be incorporated here with more work. In another direction, Conrey and Farmer [3] have shown for small N that with the additional assumption of a precise Euler product, one can do without any hypotheses on $\Lambda_f(s, \chi)$ for $q > 1$. It is an open question whether this is true in general, though Diaconu et al. [4] showed that one may combine the approaches of Weil and Conrey–Farmer to reduce the twisting set to a single suitable modulus q.

Finally, note that [1, Thm 1.1] also applies to the classical setting if one takes $F = \mathbb{Q}$ and π_∞ a discrete series or limit of discrete series representation. In a companion paper [2] we generalize the results of [1] in a different direction which also allows some poles among the twists by unramified idèle class characters. It is interesting to compare that result with Theorem 1.1. On the one hand, Theorem 1.1 is stronger in that it does not assume the existence of an Euler product and only uses analytic data from a restricted set of twists. On the other hand, the stronger hypotheses inherent in the adelic approach, namely that we start with an irreducible admissible representation, rule out the pathology of E_2 and also allow $\Lambda_f(s)$ to have more general sets of poles.

2 Preliminaries

In this section we prove a few lemmas required for the main argument. We will assume the hypotheses of Theorem 1.1 throughout, as well as some additional notation defined below.

Note that by replacing g_n by ϵg_n, we may assume without loss of generality that $\epsilon = 1$. Thus, the hypotheses of Theorem 1.1 are unchanged if we exchange f and g and replace ψ by $\overline{\psi}$, so we are free to reverse the roles of f and g in the argument. Next, let $\sigma_2 = \sigma + 1$, $\sigma_1 = k - \sigma_2$ and $K = \lfloor \sigma_2 \rfloor + 1$. Then by absolute convergence and the functional equation, the poles of $\Lambda_f(s, \chi)$ and $\Lambda_g(s, \overline{\chi})$ are contained in the strip $\Re(s) \in [\sigma_1, \sigma_2]$ for every primitive χ of modulus $q \in \mathcal{P} \cup \{1\}$.

For any $\alpha \in \mathbb{Q}^\times$ we define the additive twists

$$\Lambda_f(s, \alpha) = (2\pi|\alpha|)^{-s}\Gamma(s) \sum_{n=1}^{\infty} f_n e(n\alpha) n^{-s} \quad \text{and} \quad \Lambda_g(s, \alpha) = (2\pi|\alpha|)^{-s}\Gamma(s) \sum_{n=1}^{\infty} g_n e(n\alpha) n^{-s}$$

for $\Re(s) > \sigma_2$. If χ is a primitive character of modulus $q \in \mathcal{P}$ then we have

$$\chi(n) = \frac{\tau(\chi)}{q} \sum_a \overline{\chi}(-a) e\left(\frac{an}{q}\right),$$

where the sum runs over any set of non-zero integers a representing the residue classes mod q. Multiplying by $f_n n^{-s}$ and summing over n, we have

$$\Lambda_f(s, \chi) = \frac{\tau(\chi)}{q} \sum_a \overline{\chi}(-a) \left|\frac{a}{q}\right|^s \Lambda_f\left(s, \frac{a}{q}\right), \tag{2}$$

and similarly for g. Conversely, if $q \in \mathcal{P}$ and $(a,q)=1$ then

$$e\left(\frac{an}{q}\right) = 1 - \frac{q}{q-1}\chi_0(n) + \frac{1}{q-1}\sum_{\substack{\chi \pmod q\\ \chi\neq\chi_0}} \tau(\overline{\chi})\chi(an),$$

where χ_0 denotes the trivial character mod q; it follows that

$$\left|\frac{a}{q}\right|^s \Lambda_f\left(s,\frac{a}{q}\right) = \Lambda_f(s) - \frac{q}{q-1}\Lambda_f(s,\chi_0) + \frac{1}{q-1}\sum_{\substack{\chi \pmod q\\ \chi\neq\chi_0}} \tau(\overline{\chi})\chi(a)\Lambda_f(s,\chi).$$

Since we have not imposed any hypotheses on $\Lambda_f(s,\chi_0)$, we cannot conclude from this the meromorphic continuation of $\Lambda_f(s,\alpha)$ for an individual α. However, if $(a,q)=(a',q)=1$ then we see that

$$\left|\frac{a}{q}\right|^s \Lambda_f\left(s,\frac{a}{q}\right) - \left|\frac{a'}{q}\right|^s \Lambda_f\left(s,\frac{a'}{q}\right) = \frac{1}{q-1}\sum_{\substack{\chi \pmod q\\ \chi\neq\chi_0}} \tau(\overline{\chi})(\chi(a)-\chi(a'))\Lambda_f(s,\chi) \tag{3}$$

continues meromorphically to $\mathbb{C}$, with poles confined to the strip $\Re(s)\in[\sigma_1,\sigma_2]$, and similarly for g.

For any open interval $(a,b)\subset\mathbb{R}$ (including those with infinite endpoints), let $\mathcal{M}(a,b)$ be the set of functions which are holomorphic on $\{s\in\mathbb{C} : \Re(s)\in(a,b)\}$, except for at most simple poles at integer points, and bounded on $\{s\in\mathbb{C} : \Re(s)\in[c,d], |\Im(s)|\geq 1\}$ for each compact subinterval $[c,d]\subset(a,b)$. Let $\mathcal{H}(a,b)\subset\mathcal{M}(a,b)$ be the subset of functions which are in addition holomorphic at each integer point in (a,b). By absolute convergence and the functional equation, we see that (3) is an element of $\mathcal{H}(-\infty,\sigma_1)$.

Lemmas

With the notation in hand, our first step is to apply Hecke's argument to $\Lambda_f(s)$:

Lemma 2.1. For any $z\in\mathbb{C}$ with $\Im(z)>0$ we have

$$f(z) - f_0 - \left(-i\sqrt{N}z\right)^{-k}\left[g\left(-\frac{1}{Nz}\right) - g_0\right] = \frac{1}{2\pi i}\oint \Lambda_f(s)(-iz)^{-s}\,ds, \tag{4}$$

where the integral is taken over a circle enclosing all poles of $\Lambda_f(s)$, and we define $(-iz)^{-s} = e^{-s\log(-iz)}$ using the standard branch of the logarithm. □

Proof. Since $P(s)\Lambda_f(s)$ is entire of finite order, the Phragmén–Lindelöf convexity principle implies that $\Lambda_f(s)$ decays rapidly as $|\Im(s)|\to\infty$ in a fixed vertical strip. The identity (4) thus follows by Mellin inversion, along the lines of [8, Thm 4.3.5]. ■

Next, we consider (4) as z tends to a cusp $\beta\in\mathbb{Q}^\times$. Let us first consider the left-hand side. By a nearly identical argument to that of [1, §2], we obtain:

Lemma 2.2. Let $\alpha\in\mathbb{Q}^\times$, $\beta=-1/N\alpha$ and $z=\beta+i|\beta|y$ for some $y\in(0,1/4)$. Then

$$\begin{aligned} f(z) - f_0 &- \left(-i\sqrt{N}z\right)^{-k}\left[g\left(-\frac{1}{Nz}\right) - g_0\right]\\ &= O_{\alpha,M}\left(y^{M-K}\right) + \frac{1}{2\pi i}\int_{\Re(s)=\sigma+2} \Lambda_f(s,\beta)y^{-s}\,ds\\ &\quad - \left(-i\sqrt{N}\alpha\right)^k \sum_{m=0}^{M-1}\left(-i\,\mathrm{sgn}(\alpha)\right)^m \frac{1}{2\pi i}\int_{\Re(s)=\sigma+2}\binom{s+m-k}{m}\Lambda_g(s+m,\alpha)y^{-s}\,ds \end{aligned}$$

for any $M\in\mathbb{Z}_{\geq 0}$. □

Now let us focus on the right-hand side of (4) for z as in Lemma 2.2. Note that for $y < 1/4$ we have

$$
\begin{aligned}
(-iz)^{-s} &= e^{-i\frac{\pi}{2}\operatorname{sgn}(\alpha)s}|N\alpha|^s(1-i\operatorname{sgn}(\alpha)y)^{-s}\\
&= e^{-i\frac{\pi}{2}\operatorname{sgn}(\alpha)s}|N\alpha|^s\sum_{j=0}^{M-K-1}\binom{-s}{j}(-i\operatorname{sgn}(\alpha)y)^j + O_{\alpha,M,s}\big(y^{M-K}\big),
\end{aligned}
$$

and we may take the O-constant to be uniform for s varying in a fixed compact set. Multiplying this by $\Lambda_f(s)$ and integrating, we get

$$
\frac{1}{2\pi i}\oint \Lambda_f(s)(-iz)^{-s}\,ds = \sum_{j=0}^{M-K-1} P_j(\alpha)y^j + O_{\alpha,M}\big(y^{M-K}\big),
$$

where

$$
P_j(\alpha) = \frac{(-i\operatorname{sgn}(\alpha))^j}{2\pi i}\oint \Lambda_f(s)e^{-i\frac{\pi}{2}\operatorname{sgn}(\alpha)s}|N\alpha|^s\binom{-s}{j}\,ds.
$$

Hence, if we define

$$
\begin{aligned}
F_{\alpha,M}(y) = \frac{1}{2\pi i}&\int_{\Re(s)=\sigma+2}\Lambda_f(s,\beta)y^{-s}\,ds - \sum_{j=0}^{M-K-1}P_j(\alpha)\chi_{(0,1)}(y)y^j\\
&-\big(-i\sqrt{N}\alpha\big)^k\sum_{m=0}^{M-1}\big(-i\operatorname{sgn}(\alpha)\big)^m\frac{1}{2\pi i}\int_{\Re(s)=\sigma+2}\binom{s+m-k}{m}\Lambda_g(s+m,\alpha)y^{-s}\,ds,
\end{aligned}
$$

where $\chi_{(0,1)}(y) = 1$ if $y < 1$ and 0 otherwise, then it follows from Lemmas 2.1 and 2.2 that $F_{\alpha,M}(y) = O_{\alpha,M}\big(y^{M-K}\big)$ for $y < 1/4$. By shifting the contours to the right, it is also clear that $F_{\alpha,M}(y)$ tends rapidly to 0 as $y \to \infty$, so we may take its Mellin transform to obtain:

Lemma 2.3. Let $\alpha \in \mathbb{Q}^\times$ and $\beta = -1/N\alpha$. For any $M \in \mathbb{Z}_{\geq 0}$,

$$
\big(-i\sqrt{N}\alpha\big)^k\sum_{m=0}^{M-1}\big(-i\operatorname{sgn}(\alpha)\big)^m\binom{s+m-k}{m}\Lambda_g(s+m,\alpha) + \sum_{j=0}^{M-K-1}\frac{P_j(\alpha)}{s+j} - \Lambda_f(s,\beta)
$$

continues to an element of $\mathcal{H}(K-M,\infty)$. □

Next, suppose that $\alpha = u/v$ with $(u,v) = 1$ and $v > 0$. We define

$$
T_\alpha = \left\{\frac{p}{u} : p \in \mathcal{P},\ p \equiv u \pmod{v}\right\}.
$$

Note that T_α is an infinite set, by hypothesis. If $\lambda \in T_\alpha$ then $\lambda\alpha = p/v$, where $p \equiv u \pmod{v}$, and it follows that $\Lambda_g(s,\lambda\alpha) = |\lambda|^{-s}\Lambda_g(s,\alpha)$. Replacing α by $\lambda\alpha$ in Lemma 2.3 and multiplying by $|\lambda\beta|^s = |\lambda^{-1}N\alpha|^{-s}$, we see that

$$
\begin{aligned}
&\sum_{m=0}^{M-1}\lambda^{k-m}N^{-k/2}(-iN\alpha)^{k+m}|N\alpha|^{-s-m}\binom{s+m-k}{m}\Lambda_g(s+m,\alpha)\\
&+|\lambda\beta|^s\left(\sum_{j=0}^{M-K-1}\frac{P_j(\lambda\alpha)}{s+j} - \Lambda_f\big(s,\lambda^{-1}\beta\big)\right)
\end{aligned}
\tag{5}
$$

is in $\mathcal{H}(K-M,\infty)$.

Now, we would like to average over different choices of λ to isolate a single term in the above sum. To that end, choose an integer $m_0 \geq 0$ and take $M > m_0$. From the Vandermonde determinant we see that for any subset $T_{\alpha,M} \subset T_\alpha$ of cardinality M, there are numbers $c_\lambda \in \mathbb{Q}$ for $\lambda \in T_{\alpha,M}$ such that

$$
\sum_{\lambda\in T_{\alpha,M}} c_\lambda\lambda^{k-m} = \begin{cases}1 & \text{if } m = m_0,\\ 0 & \text{if } m \neq m_0\end{cases}\qquad\text{for all } m \in \mathbb{Z}\cap[0,M).
\tag{6}
$$

Summing (5) (after scaling by c_λ) over $\lambda \in T_{\alpha,M}$ and replacing s by $s - m_0$, we obtain:

Lemma 2.4. Let $\alpha \in \mathbb{Q}^\times$, $\beta = -1/N\alpha$, $m_0 \in \mathbb{Z}_{\geq 0}$ and $M \in \mathbb{Z}$ with $M > m_0$. Let $T_{\alpha,M}$ be an arbitrary subset of T_α of cardinality M, and let c_λ for $\lambda \in T_{\alpha,M}$ be defined by (6). Then

$$\begin{aligned} &N^{-k/2}(-iN\alpha)^{k+m_0}|N\alpha|^{-s}\binom{s-k}{m_0}\Lambda_g(s,\alpha) \\ &\qquad + \sum_{\lambda \in T_{\alpha,M}} c_\lambda |\lambda\beta|^{s-m_0}\left(\sum_{j=0}^{M-K-1}\frac{P_j(\lambda\alpha)}{s-m_0+j} - \Lambda_f\big(s-m_0,\lambda^{-1}\beta\big)\right) \end{aligned} \tag{7}$$

continues to an element of $\mathcal{H}(m_0+K-M,\infty)$. In particular,

$$N^{-k/2}(-iN\alpha)^{k+m_0}|N\alpha|^{-s}\binom{s-k}{m_0}\Lambda_g(s,\alpha) - \sum_{\lambda \in T_{\alpha,M}} c_\lambda|\lambda\beta|^{s-m_0}\Lambda_f\big(s-m_0,\lambda^{-1}\beta\big) \tag{8}$$

continues to an element of $\mathcal{M}(m_0+K-M,\infty)$. □

3 Proof of Theorem 1.1

We now have the necessary preliminaries to prove our main theorem. Let $q, q' \in \mathcal{P} \cup \{1\}$ with $q \neq q'$, and $a \in \mathbb{Z}$ with $|a| \in \mathcal{P} \cup \{1\}$ and $(a,q) = (a,q') = 1$. In what follows, we will apply Lemma 2.4 variously with $\alpha = a/Nq$, a/Nq' and a/q.

First, we consider the difference between (8) evaluated at $\alpha = a/Nq$ and $\alpha' = a/Nq'$. Note that since α and α' have the same numerator, $T_\alpha \cap T_{\alpha'}$ is infinite, and thus we may take $T_{\alpha,M} = T_{\alpha',M}$. Correspondingly, $\lambda^{-1}\beta$ (resp. $\lambda^{-1}\beta'$) is of the form $-q/p$ (resp. $-q'/p$) for $p \in \mathcal{P}$, $p \nmid qq'$, and it follows from (3) that

$$|\beta|^{s-m_0}\Lambda_f\big(s-m_0,\lambda^{-1}\beta\big) - |\beta'|^{s-m_0}\Lambda_f\big(s-m_0,\lambda^{-1}\beta'\big)$$

is in $\mathcal{H}(-\infty, m_0+\sigma_1)$. Therefore, by Lemma 2.4,

$$\alpha^{k+m_0}|\alpha|^{-s}\binom{s-k}{m_0}\Lambda_g(s,\alpha) - (\alpha')^{k+m_0}|\alpha'|^{-s}\binom{s-k}{m_0}\Lambda_g(s,\alpha')$$

is in $\mathcal{M}(m_0+K-M, m_0+\sigma_1)$. Choosing $m_0 \geq \sigma_2$ and M arbitrarily large, we see that

$$\alpha^{k+m_0}|\alpha|^{-s}\Lambda_g(s,\alpha) - (\alpha')^{k+m_0}|\alpha'|^{-s}\Lambda_g(s,\alpha')$$

is in $\mathcal{M}(-\infty,k)$. Finally, since $\alpha \neq \alpha'$ and m_0 is arbitrary, we see that $\Lambda_g(s,\alpha)$ is in $\mathcal{M}(-\infty,k)$. Reversing the roles of f and g, we get the same conclusion for $\Lambda_f(s,\alpha)$.

Next, consider $\alpha = a/q$ in (8). Then $\lambda^{-1}\beta$ is of the form $-q/Np$ for $p \in \mathcal{P}$, $(q,p) = 1$, so that by the above, $\Lambda_f(s,\lambda^{-1}\beta)$ is in $\mathcal{M}(-\infty,k)$. Taking $m_0 = 0$ and M arbitrarily large, we again find that $\Lambda_g(s,\alpha)$ is in $\mathcal{M}(-\infty,k)$, and similarly for $\Lambda_f(s,\alpha)$.

To summarize, we have established that for $\alpha = a/q, a/Nq$, both $\Lambda_f(s,\alpha)$ and $\Lambda_g(s,\alpha)$ belong to $\mathcal{M}(-\infty,k)$. In particular, taking $\alpha = 1$, we get this conclusion for $\Lambda_f(s)$ and $\Lambda_g(s)$; taking account of the functional equation as well, we see that $\Lambda_f(s)$ has at most simple poles at integer points in $[\sigma_1,\sigma_2]$. Thus, by the residue theorem, for any $\alpha \in \mathbb{Q}^\times$ we have

$$\begin{aligned} P_j(\alpha) &= \sum_{\ell \in \mathbb{Z}\cap[\sigma_1,\sigma_2]} \operatorname{Res}_{s=k-\ell}\Lambda_f(s)(-i\operatorname{sgn}(\alpha))^{j+k-\ell}|N\alpha|^{k-\ell}\binom{\ell-k}{j} \\ &= -N^{k/2}|\alpha|^{-j}\sum_{\ell \in \mathbb{Z}\cap[\sigma_1,\sigma_2]} \operatorname{Res}_{s=\ell}\Lambda_g(s)(-i\alpha)^{k+j-\ell}\binom{\ell-k}{j}. \end{aligned}$$

This yields the following:

Lemma 3.1. Let notation be as in Lemma 2.4, and suppose that $\Lambda_g(s,\alpha)$ continues to an element of $\mathcal{M}(-\infty,k)$, as does $\Lambda_f(s,\lambda^{-1}\beta)$ for every $\lambda \in T_\alpha$. Let $s_0 \in \mathbb{Z}$ with $s_0 < k$. If $m_0 \geq \sigma_2$ and $M > m_0 + \max(0, K-s_0)$ then

$$\begin{aligned} &\operatorname{Res}_{s=s_0}\sum_{\lambda \in T_{\alpha,M}} c_\lambda|\lambda\beta|^{s-m_0}\Lambda_f\big(s-m_0,\lambda^{-1}\beta\big) \\ &= N^{-k/2}\big(-iN\alpha\big)^{k+m_0}|N\alpha|^{-s_0}\left[\binom{s_0-k}{m_0}\operatorname{Res}_{s=s_0}\Lambda_g(s,\alpha) - (-|\alpha|)^{-s_0}\binom{s_0-k}{m_0-s_0}\operatorname{Res}_{s=s_0}\Lambda_g(s)\right]. \end{aligned} \tag{9}$$

□

Proof. By our assumption on M, we have $K-M<s_0-m_0<k-m_0$. We substitute the expression for $P_j(\alpha)$ from above into (7) and consider the residue at $s=s_0$:

$$\begin{aligned}&\operatorname{Res}_{s=s_0}\sum_{\lambda\in T_{\alpha,M}}c_\lambda|\lambda\beta|^{s-m_0}\Lambda_f\big(s-m_0,\lambda^{-1}\beta\big)\\&=N^{-k/2}(-iN\alpha)^{k+m_0}|N\alpha|^{-s_0}\Bigg[\binom{s_0-k}{m_0}\operatorname{Res}_{s=s_0}\Lambda_g(s,\alpha)\\&-(-|\alpha|)^{-s_0}\sum_{\ell\in\mathbb{Z}\cap[\sigma_1,\sigma_2]}\sum_{\lambda\in T_{\alpha,M}}c_\lambda\lambda^{k+s_0-\ell-m_0}\operatorname{Res}_{s=\ell}\Lambda_g(s)(-i\alpha)^{s_0-\ell}\binom{\ell-k}{m_0-s_0}\Bigg].\end{aligned}$$

Note that $0>s_0-\ell-m_0>-M$ for every $\ell\in[\sigma_1,\sigma_2]\cap\mathbb{Z}$. Thus, by (6), the sum over λ isolates the single term $\ell=s_0$ and the lemma follows. ∎

Let us again consider $\alpha=a/Nq$ and $\alpha'=a/Nq'$ and take the difference between (9) evaluated at these points. Note that the hypotheses of Lemma 3.1 are satisfied. If we take $m_0\geq\sigma_2$ and $s_0<\sigma_1$ then $|\beta|^{s-m_0}\Lambda_f(s-m_0,\lambda^{-1}\beta)-|\beta'|^{s-m_0}\Lambda_f(s-m_0,\lambda^{-1}\beta')$ is holomorphic at s_0, as is $\Lambda_g(s)$. Hence, we obtain

$$\alpha^{k+m_0}|\alpha|^{-s_0}\operatorname{Res}_{s=s_0}\Lambda_g(s,\alpha)=(\alpha')^{k+m_0}|\alpha'|^{-s_0}\operatorname{Res}_{s=s_0}\Lambda_g(s,\alpha').$$

Since $\alpha\neq\alpha'$ and m_0 is arbitrary, we must have $\operatorname{Res}_{s=s_0}\Lambda_g(s,\alpha)=0$. Therefore, $\Lambda_g(s,\alpha)$ is in $\mathcal{H}(-\infty,\sigma_1)$, and similarly for $\Lambda_f(s,\alpha)$.

Finally, we apply Lemma 3.1 once more with $\alpha=a/q$. If we take $m_0\geq\sigma_2$, then by the above the left-hand side of (9) is 0 for any $s_0<k$. Hence, we have

$$\binom{s_0-k}{m_0}\operatorname{Res}_{s=s_0}\Lambda_g(s,\alpha)=(-|\alpha|)^{-s_0}\binom{s_0-k}{m_0-s_0}\operatorname{Res}_{s=s_0}\Lambda_g(s).\tag{10}$$

In particular, taking $\alpha=1$, we find

$$\left[\binom{s_0-k}{m_0}-(-1)^{s_0}\binom{s_0-k}{m_0-s_0}\right]\operatorname{Res}_{s=s_0}\Lambda_g(s)=0$$

for any $s_0<k$, $m_0\geq\sigma_2$.

Since we may take m_0 arbitrarily large, it is not hard to see that we may arrange for the factor in brackets to be non-zero unless $s_0=0$ or $s_0=k-1$. Again swapping the roles of f and g and reasoning with the functional equation, we conclude that $\Lambda_f(s)$ and $\Lambda_g(s)$ can only have poles at $s\in\{0,k\}$ for $k\neq2$, and $s\in\{0,1,2\}$ for $k=2$.

Moreover, for $s_0<k$ we see from (10) that $|\alpha|^{s_0}\operatorname{Res}_{s=s_0}\Lambda_g(s,\alpha)$ is independent of α. Thus, if χ is a non-trivial Dirichlet character of modulus $q\in\mathcal{P}$, we see from (2) and the functional equation that $\Lambda_f(s,\chi)$ and $\Lambda_g(s,\overline{\chi})$ are in $\mathcal{H}(-\infty,\infty)$, i.e. they are entire and bounded in vertical strips.

Conclusion of the proof

If $k\neq2$ or if $k=2$ and $\Lambda_f(s)$ is holomorphic at $s=1$ then the conclusion of Theorem 1.1 follows from Weil's theorem. To conclude the remaining case, let us write

$$f^*(z)=f(z)-cE_2(z),\quad g^*(z)=g(z)+cNE_2(Nz),$$

where $c=\frac{\pi}{6}\operatorname{Res}_{s=1}\Lambda_f(s)$, and let $\{f_n^*\}_{n=1}^\infty$, $\{g_n^*\}_{n=1}^\infty$ be the corresponding sequences of Fourier coefficients. Starting from the formula

$$\Lambda_{E_2}(s)=-24(2\pi)^{-s}\Gamma(s)\zeta(s)\zeta(s-1)$$

we check that $\Lambda_{f^*}(s)$ is holomorphic at $s=1$, and if ψ is the trivial character modulo N then the functional equation (1) remains true with f,g replaced by f^*,g^*. The conclusion follows from Weil's theorem in this case.

For non-trivial ψ, we only obtain (1) for $q\equiv1\pmod N$ in general. From the proof of Weil's theorem (see, e.g., [8, Lemma 4.3.14]), this is enough to see that $f^*\in M_2(\Gamma_1(N))$. It follows that we may write $f^*=\sum_{\xi\pmod N}f_\xi$, where the sum runs over all Dirichlet characters ξ modulo N and $f_\xi\in M_2(\Gamma_0(N),\xi)$. Set $g_\xi(z)=-\frac{1}{Nz^2}f_\xi\big(-\frac{1}{Nz}\big)$, so that $g_\xi\in M_2(\Gamma_0(N),\overline{\xi})$ and $g^*=\sum_{\xi\pmod N}g_\xi$. Note also that (1) is satisfied with

f, g replaced by f_ξ, g_ξ and ψ replaced by ξ. Thus, using the functional equations for $\Lambda_f(s,\chi)$, $\Lambda_{E_2}(s,\chi)$ and $\Lambda_{f_\xi}(s,\chi)$, we see that

$$\overline{\psi}(q)\Lambda_f(s,\chi) - c\Lambda_{E_2}(s,\chi) - \sum_{\xi \pmod N} \overline{\xi}(q)\Lambda_{f_\xi}(s,\chi) = 0 \tag{11}$$

identically for any non-trivial character χ of modulus $q \in \mathcal{P}$. Hence, the Dirichlet coefficients of (11) must vanish identically. Since we may take q arbitrarily large in any fixed invertible residue class $a \pmod N$, this implies that

$$\overline{\psi}(a)\Lambda_f(s) - c\Lambda_{E_2}(s) - \sum_{\xi \pmod N} \overline{\xi}(a)\Lambda_{f_\xi}(s) = 0$$

for all a with $(a, N) = 1$. Averaging this equation over a and using that ψ is non-trivial, we deduce that $\Lambda_{f_{\xi_0}}(s) = -c\Lambda_{E_2}(s)$, where ξ_0 is the trivial character modulo N. Since $\Lambda_{E_2}(s)$ has a pole at $s = 1$ but $\Lambda_{f_{\xi_0}}(s)$ does not, this is only possible if $c = 0$, and the result follows.

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